Discrete Applied Mathematics 159 (2011) 224–231

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Partitioning a graph into offensive k-alliances

José M. Sigarreta [a], Ismael G. Yero [b], Sergio Bermudo [c], Juan A. Rodríguez-Velázquez [b,*]

[a] Faculty of Mathematics, Autonomous University of Guerrero, Carlos E. Adame 5, Col. La Garita, Acapulco, Guerrero, Mexico
[b] Departament d'Enginyeria Informàtica i Matemàtiques, Universitat Rovira i Virgili, Av. Països Catalans 26, 43007 Tarragona, Spain
[c] Department of Economy, Quantitative Methods and Economic History, Pablo de Olavide University, Carretera de Utrera Km. 1, 41013-Sevilla, Spain

ARTICLE INFO

Article history:
Received 29 April 2009
Received in revised form 15 August 2010
Accepted 9 November 2010
Available online 3 December 2010

Keywords:
Offensive alliances
Dominating sets
Domination number
Chromatic number

ABSTRACT

An offensive k-alliance in a graph is a set S of vertices with the property that every vertex in the boundary of S has at least k more neighbors in S than it has outside of S. An offensive k-alliance S is called global if it forms a dominating set. In this paper we study the problem of partitioning the vertex set of a graph into (global) offensive k-alliances. The (global) offensive k-alliance partition number of a graph $\Gamma = (V, E)$, denoted by $(\psi_k^{go}(\Gamma))$ $\psi_k^{o}(\Gamma)$, is defined to be the maximum number of sets in a partition of V such that each set is an offensive (a global offensive) k-alliance. We show that $2 \le \psi_k^{go}(\Gamma) \le 3 - k$ if Γ is a graph without isolated vertices and $k \in \{2 - \Delta, \ldots, 0\}$. Moreover, we show that if Γ is partitionable into global offensive k-alliances for $k \ge 1$, then $\psi_k^{go}(\Gamma) = 2$. As a consequence of the study we show that the chromatic number of Γ is at most 3 if $\psi_0^{go}(\Gamma) = 3$. In addition, for $k \le 0$, we obtain tight bounds on $\psi_k^{o}(\Gamma)$ and $\psi_k^{go}(\Gamma)$ in terms of several parameters of the graph including the order, size, minimum and maximum degree. Finally, we study the particular case of the cartesian product of graphs, showing that $\psi_k^{o}(\Gamma_1 \times \Gamma_2) \ge \psi_{k_1}^{o}(\Gamma_1)\psi_{k_2}^{o}(\Gamma_2)$, for $k \le \min\{k_1 - \Delta_2, k_2 - \Delta_1\}$, where Δ_i denotes the maximum degree of Γ_i, and $\psi_k^{go}(\Gamma_1 \times \Gamma_2) \ge \max\{\psi_{k_1}^{go}(\Gamma_1), \psi_{k_2}^{go}(\Gamma_2)\}$, for $k \le \min\{k_1, k_2\}$.

1. Introduction

Since (defensive, offensive and powerful) alliances in graphs were first introduced by Kristiansen et al. [10], several authors have studied their mathematical properties [1–18]. We are interested in the study of offensive k-alliances [1,3,6,7,18]. We focus our attention on the problem of partitioning the vertex set of a graph into (global) offensive k-alliances. This problem has been previously studied, for the case of defensive k-alliances, by Shafique and Dutton [14,15] and the particular case $k = -1$ has been studied by Eroh and Gera [4,5] and by Haynes and Lachniet [9].

We begin by stating the terminology used. Throughout this article, $\Gamma = (V, E)$ denotes a simple graph of order $|V| = n$ and size $|E| = m$. We denote two adjacent vertices u and v by $u \sim v$, the degree of a vertex $v \in V$ by $\delta(v)$, the minimum degree by δ and the maximum degree by Δ. For a nonempty set $S \subseteq V$, and a vertex $v \in V$, $N_S(v)$ denotes the set of neighbors that v has in S: $N_S(v) := \{u \in S : u \sim v\}$, and the degree of v in S will be denoted by $\delta_S(v) = |N_S(v)|$. For a set $S \subset V$, we denote the complement of S in V by $\overline{S}$ and the boundary of S is defined as $\partial(S) := \bigcup_{v \in S} N_{\overline{S}}(v)$.

A set $S \subseteq V$ is a *dominating set* in Γ if for every vertex $v \in \overline{S}$, $\delta_S(v) > 0$ (every vertex in $\overline{S}$ is adjacent to at least one vertex in S). The *domination number* of Γ, denoted by $\gamma(\Gamma)$, is the minimum cardinality of a dominating set in Γ. For $k \in \{2 - \Delta, \ldots, \Delta\}$, a nonempty set $S \subseteq V$ is an *offensive k-alliance* in Γ if

$$\delta_S(v) \ge \delta_{\overline{S}}(v) + k, \quad \forall v \in \partial(S) \tag{1}$$

* Corresponding author.
E-mail addresses: josemariasigarretaalmira@hotmail.com (J.M. Sigarreta), ismael.gonzalez@urv.cat (I.G. Yero), sbernav@upo.es (S. Bermudo), juanalberto.rodriguez@urv.cat (J.A. Rodríguez-Velázquez).

0166-218X/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.dam.2010.11.005

or, equivalently,

$$\delta(v) \geq 2\delta_{\overline{S}}(v) + k, \quad \forall v \in \partial(S).$$

It is clear that if $k > \Delta$, no set S satisfies (1) and, if $k < 2 - \Delta$, all the subsets of V satisfy it. An offensive k-alliance S is called *global* if it is a dominating set. The *offensive k-alliance number* of Γ, denoted by $a_k^o(\Gamma)$, is defined as the minimum cardinality of an offensive k-alliance in Γ. Analogously, the *global offensive k-alliance number* of Γ, denoted by $\gamma_k^o(\Gamma)$, is defined as the minimum cardinality of a global offensive k-alliance in Γ. Notice that $\gamma_k^o(\Gamma) \geq a_k^o(\Gamma)$ and $\gamma_{k+1}^o(\Gamma) \geq \gamma_k^o(\Gamma) \geq \gamma(\Gamma)$ for every k. The *offensive k-alliance partition number* of Γ, $\psi_k^o(\Gamma)$, $k \in \{2 - \Delta, \ldots, \Delta\}$, is defined as the maximum number of sets in a partition of V such that each set is an offensive k-alliance.

If V can be partitioned into global offensive k-alliances, then there exist a global offensive k-alliance S and a vertex of minimum degree v such that $v \notin S$ and $\delta = \delta(v) \geq 2\delta_{\overline{S}}(v) + k$. Therefore, if $k > \delta$, then V cannot be partitioned into global offensive k-alliances. The *global offensive k-alliance partition number* of Γ, $\psi_k^{go}(\Gamma)$, $k \in \{2 - \Delta, \ldots, \delta\}$, is defined as the maximum number of sets in a partition of V such that each set is a global offensive k-alliance. Hereafter we will say that $\Pi_r(\Gamma)$ is a partition of Γ into r (global) offensive k-alliances.

Notice that if every vertex of Γ has even degree and k is odd, or every vertex of Γ has odd degree and k is even, then every (global) offensive k-alliance in Γ is an offensive (a global offensive) $(k+1)$-alliance and vice versa. Hence, in such a case, $a_k^o(\Gamma) = a_{k+1}^o(\Gamma)$, $\gamma_k^o(\Gamma) = \gamma_{k+1}^o(\Gamma)$, $\psi_k^o(\Gamma) = \psi_{k+1}^o(\Gamma)$ and $\psi_k^{go}(\Gamma) = \psi_{k+1}^{go}(\Gamma)$.

We now introduce an important class of graphs that will provide useful examples in the later sections. Let $\mathbb{Z}_n$ be the additive group of integers modulo n and let $M \subset \mathbb{Z}_n$, such that $i \in M$ if and only if $-i \in M$. We can construct a graph $\Gamma = (V, E)$ as follows, the vertices of V are the elements of $\mathbb{Z}_n$ and (i, j) is an edge in E if and only if $j - i \in M$. This graph is called a *circulant of order n* and we will denote it by $CR(n, M)$. With this notation, a cycle graph is $CR(n, \{-1, 1\})$ and the complete graph is $CR(n, \mathbb{Z}_n)$. In order to simplify the notation, we will use $CR(n, t)$, $0 < t \leq \frac{n}{2}$, instead of $CR(n, \{-t, -t+1, \ldots, -1, 1, 2, \ldots, t\})$. We emphasize that $CR(n, t)$ is a $(2t)$-regular graph. Note that, if n is even, $\Pi_2(CR(n, 2)) = \{\{1, 3, 5, \ldots, n-1\}, \{2, 4, 6, \ldots, n\}\}$ is a partition of $CR(n, 2)$ into global offensive 0-alliances; moreover, if $n = 4j$, $\Pi_4(CR(n, 2)) = \{\{1, 5, \ldots, n-3\}, \{2, 6, \ldots, n-2\}, \{3, 7, \ldots, n-1\}, \{4, 8, \ldots, n\}\}$ is a partition of $CR(n, 2)$ into global offensive (-2)-alliances.

We say that a graph Γ is partitionable into (global) offensive k-alliances if ($\psi_k^{go}(\Gamma) \geq 2$) $\psi_k^o(\Gamma) \geq 2$.

2. Partitioning a graph into offensive k-alliances

Proposition 1. *For any graph Γ without isolated vertices, there exists $k \in \{0, \ldots, \delta\}$ such that Γ is partitionable into global offensive k-alliances.*

Proof. If $\delta \geq 1$ and $\{X, Y\}$ is a partition of V such that the edge cut-set between X and Y has maximum cardinality, then X and Y are dominating sets. Moreover, for every $x_i \in X$ there exists $k_i \in \mathbb{Z}$, $k_i \geq 0$, such that $\delta_Y(x_i) = \delta_X(x_i) + k_i$. Taking $k = \min_{x_i \in X}\{k_i\}$, then we have that Y is a global offensive k-alliance in Γ. Analogously we obtain that there exists $r \in \mathbb{Z}$, $r \geq 0$, such that X is a global offensive r-alliance in Γ. Therefore, taking $t = \min\{k, r\}$ we conclude that $\{X, Y\}$ is a partition of V into two global offensive t-alliances in Γ. □

Corollary 2. *Any graph without isolated vertices is partitionable into global offensive* 0*-alliances.*

Theorem 3. *If a graph is partitionable into $r \geq 3$ global offensive k-alliances, then $k \leq 3 - r$.*

Proof. We suppose that $\Pi_r(\Gamma) = \{S_1, \ldots, S_r\}$ is a partition of Γ into $r \geq 3$ global offensive k-alliances. For every $v \in S_r$ we have

$$\begin{aligned}
\delta_{S_1}(v) &\geq \delta_{\overline{S_1}}(v) + k \geq \sum_{j=2}^{r-1} \delta_{S_j}(v) + k \\
&\geq \sum_{j=2}^{r-1} (\delta_{\overline{S_j}}(v) + k) + k \\
&\geq \sum_{j=2}^{r-1} \sum_{i=1; i\neq j}^{r-1} \delta_{S_i}(v) + \sum_{j=2}^{r-1} k + k \\
&= \sum_{j=2}^{r-1} \delta_{S_1}(v) + \sum_{j=2}^{r-1} \sum_{i=2; i\neq j}^{r-1} \delta_{S_i}(v) + k(r-1) \\
&\geq (r-2)\delta_{S_1}(v) + \sum_{j=2}^{r-1} \sum_{i=2; i\neq j}^{r-1} 1 + k(r-1) \\
&= (r-2)\delta_{S_1}(v) + (r-2)(r-3) + k(r-1).
\end{aligned}$$

Therefore,

$$\begin{aligned} 0 &\geq (r-3)\delta_{S_1}(v) + (r-2)(r-3) + k(r-1) \\ &\geq (r-3) + (r-2)(r-3) + k(r-1) \\ &= (r-1)(r-3+k), \end{aligned}$$

that is, $k \leq 3 - r$. $\square$

From Theorem 3 we have that if a graph is partitionable into $r \geq 3$ global offensive k-alliances, then $k \leq 0$, so we obtain the following interesting consequence.

Corollary 4. *If Γ is partitionable into global offensive k-alliances for $k \geq 1$, then $\psi_k^{go}(\Gamma) = 2$.*

From Corollary 2 we have that any graph without isolated vertices is partitionable into global offensive 0-alliances. In consequence, from Theorem 3 we obtain the following result.

Corollary 5. *Let Γ be a graph without isolated vertices. If $k \in \{2-\Delta, \ldots, 0\}$, then $2 \leq \psi_k^{go}(\Gamma) \leq 3 - k$.*

An example of a graph where $\psi_0^{go}(\Gamma) = 2$ is the complete graph $\Gamma = K_n$ and an example of a graph where $\psi_0^{go}(\Gamma) = 3$ is the cycle graph C_{3t}, $t \geq 1$.

2.1. Global offensive k-alliance partition number and chromatic number

In this section, motivated by Corollary 5, we will study the cases $\psi_0^{go}(\Gamma) = 2$ and $\psi_0^{go}(\Gamma) = 3$ and, as a consequence of the study, we shall show the relationship that exists among the chromatic number of Γ, $\chi(\Gamma)$, and $\psi_0^{go}(\Gamma)$.

We recall that, given a positive integer t, a t-dependent set in Γ is a subset of V such that no vertex in the subset is adjacent to more than t vertices of the subset. A 0-dependent set in Γ is simply an independent set of vertices in Γ.

Theorem 6. *Any set belonging to a partition of a graph into $r \geq 3$ global offensive k-alliances is a $(-k)$-dependent[1] set.*

Proof. Let $\Pi_r(\Gamma) = \{S_1, \ldots, S_r\}$ be a partition of Γ into $r \geq 3$ global offensive k-alliances. For every $v \in S_r$,

$$\begin{aligned} \delta_{S_1}(v) &\geq \delta_{\overline{S_1}}(v) + k \geq \delta_{S_2}(v) + \delta_{S_r}(v) + k \\ &\geq \delta_{\overline{S_2}}(v) + \delta_{S_r}(v) + 2k \geq \delta_{S_1}(v) + 2\delta_{S_r}(v) + 2k. \end{aligned}$$

Therefore, $\delta_{S_r}(v) \leq -k$ and, as a consequence, S_r is a $(-k)$-dependent set. Analogously we obtain that S_i, $1 \leq i \leq r-1$, is a $(-k)$-dependent set too. $\square$

Notice that, if $k = 0$ in the above result, then $r = 3$ and as a consequence, every set in a partition into three global offensive 0-alliances is an independent set, so it leads to the following result.

Corollary 7. *If $\psi_0^{go}(\Gamma) = 3$, then $\chi(\Gamma) \leq 3$.*

A trivial example of a graph where $\psi_0^{go}(\Gamma) = 3$ and $\chi(\Gamma) = 3$ is the cycle graph C_3 and an example of a graph where $\psi_0^{go}(\Gamma) = 3$ and $\chi(\Gamma) = 2$ is the cycle graph $\Gamma = C_6$.

Remark 8. If Γ is a non-bipartite graph and $\psi_0^{go}(\Gamma) = 3$, then $\chi(\Gamma) = 3$.

An example of a graph where $\chi(\Gamma) > 3$ and $\psi_0^{go}(\Gamma) = 2$ is the complete graph $\Gamma = K_n$ with $n \geq 4$.

Corollary 9. *For any graph Γ without isolated vertices and chromatic number greater than 3, $\psi_0^{go}(\Gamma) = 2$.*

Let us see another sufficient condition for the global offensive 0-alliance number to be 2.

Theorem 10. *For any graph Γ without isolated vertices containing a vertex of odd degree, it is satisfied that $\psi_0^{go}(\Gamma) = 2$.*

Proof. By Corollary 2 and Corollary 5 we have $2 \leq \psi_0^{go}(\Gamma) \leq 3$. Let us suppose $\{S_1, S_2, S_3\}$ is a partition of Γ into global offensive 0-alliances. Without loss of generality, let us suppose S_1 contains a vertex v of odd degree. From Theorem 6 we have $\delta_{S_1}(v) = 0$. As S_2 and S_3 are global offensive 0-alliances, we obtain $\delta_{S_2}(v) \geq \delta_{\overline{S_2}}(v) = \delta_{S_3}(v) \geq \delta_{\overline{S_3}}(v) = \delta_{S_2}(v)$; in consequence, $\delta(v) = \delta_{S_2}(v) + \delta_{S_3}(v) = 2\delta_{S_2}(v)$, a contradiction. $\square$

Note that Theorem 10 is equivalent to saying that if $\psi_0^{go}(\Gamma) = 3$, then every vertex in Γ has even degree. As a consequence, for k odd, every partition of Γ into (global) offensive k-alliances is a partition of Γ into (global) offensive $(k+1)$-alliances and vice versa.

Corollary 11. *If $\psi_0^{go}(\Gamma) = 3$ and k is odd, then $a_k^o(\Gamma) = a_{k+1}^o(\Gamma)$, $\gamma_k^o(\Gamma) = \gamma_{k+1}^o(\Gamma)$, $\psi_k^o(\Gamma) = \psi_{k+1}^o(\Gamma)$ and $\psi_k^{go}(\Gamma) = \psi_{k+1}^{go}(\Gamma)$.*

[1] We recall that, by Theorem 3, if $r \geq 3$, then $k \leq 0$.

2.2. Bounds on $\psi_k^o(\Gamma)$ and $\psi_k^{go}(\Gamma)$

From the following relation between the offensive k-alliance number and the offensive k-alliance partition number, we obtain that lower bounds on $a_k^o(\Gamma)$ lead to upper bounds on $\psi_k^o(\Gamma)$:

$$a_k^o(\Gamma)\psi_k^o(\Gamma) \le n.$$

It was shown in [7] that $a_k^o(\Gamma) \ge \lceil \frac{\delta+k}{2} \rceil$; hence

$$\psi_k^o(\Gamma) \le \begin{cases} \left\lfloor \dfrac{2n}{\delta+k} \right\rfloor, & \delta+k \quad \text{even} \\ \left\lfloor \dfrac{2n}{\delta+k+1} \right\rfloor, & \delta+k \quad \text{odd.} \end{cases}$$

This bound is attained, for instance, for every δ-regular graph, $\delta \ge 1$, by taking $k = 2-\delta$. In such a case, each vertex is an offensive $(2-\delta)$-alliance and $\psi_k^o(\Gamma) = n$. Another example is $\Gamma = CR(8, 2)$ where $\{1, 2, 5, 6\}$ and $\{3, 4, 7, 8\}$ are (global) offensive 2-alliances and the above bound leads to $\psi_2^o(\Gamma) \le 2$.

Analogously, lower bounds on $\gamma_k^o(\Gamma)$ lead to upper bounds on $\psi_k^{go}(\Gamma)$:

$$\gamma_k^o(\Gamma)\psi_k^{go}(\Gamma) \le n.$$

For instance, it was shown in [1] that $\gamma_k^o(\Gamma) \ge \lceil \frac{2m+kn}{3\Delta+k} \rceil$; hence

$$\psi_k^{go}(\Gamma) \le \left\lfloor \frac{n}{\lceil \frac{2m+kn}{3\Delta+k} \rceil} \right\rfloor.$$

This bound is attained, for instance, for the circulant graph $CR(n, 2)$ for $k = -2$ and, if $n = 3j$, it is also attained for $k \in \{-1, 0\}$. In this example, as with all examples in this article using $\psi_k^{go}(CR(n, 2))$, the reader is referred to the Appendix for the proof.

Theorem 12. *If a graph Γ is partitionable into global offensive k-alliances, then*

(i) $\psi_k^{go}(\Gamma) \le \left\lfloor \frac{2m-n(k-4)}{2n} \right\rfloor$,
(ii) $\psi_k^{go}(\Gamma) \le \left\lfloor \frac{\delta-k+4}{2} \right\rfloor$,
(iii) $\psi_k^{go}(\Gamma) \le \left\lfloor \frac{4-k+\sqrt{k^2+2(\delta-k)}}{2} \right\rfloor$.

Proof. Let $\Pi_r(\Gamma) = \{S_1, S_2, \ldots, S_r\}$ be a partition of Γ into global offensive k-alliances. Since S_i is a dominating set for every $i \in \{1, \ldots, r\}$, we have that for every $v \in \overline{S_i}$, $\delta_{\overline{S_i}}(v) \ge r-2$. Thus, the bounds are obtained as follows.

(i) $\delta(v) - (r-2) \ge \delta_{S_i}(v) \ge \delta_{\overline{S_i}}(v) + k \ge r-2+k$, so $2m = \sum_{v\in V} \delta(v) \ge n(2r-4+k)$. Hence, the bound follows.
(ii) If v is a vertex of minimum degree δ, there exists $S_i \in \Pi_r(\Gamma)$ such that $v \notin S_i$; thus, $\delta = \delta(v) \ge 2\delta_{\overline{S_i}}(v)+k \ge 2(r-2)+k$.
(iii) As above, if v is a vertex of minimum degree δ, there exists $S_i \in \Pi_r(\Gamma)$ such that $v \in S_i$; thus, for every $j \ne i$, $\delta = \delta(v) \ge 2\delta_{\overline{S_j}}(v) + k \ge 2\sum_{l\ne i,j} \delta_{S_l}(v) + k$. Also, as each offensive k-alliance belonging to $\Pi_r(\Gamma)$ is a dominating set, $\delta_{S_l}(v) \ge \delta_{\overline{S_l}}(v) + k \ge r-2+k$. So we obtain $\delta \ge 2(r-2)(r-2+k)+k = 2r^2+2(k-4)r-3k+8$ and, as a consequence, $2r^2+2(k-4)r-3k+8-\delta \le 0$. Therefore $r \le \frac{4-k+\sqrt{k^2+2(\delta-k)}}{2}$. $\square$

In order to compare (ii) and (iii) for $\delta \ge 1$, we note that

$$\frac{\delta-k+4}{2} < \frac{4-k+\sqrt{k^2+2(\delta-k)}}{2}$$

if and only if $k < 2-\delta$. Examples of equality in above theorem include the following ones. Bound (i) is attained for the cycle graph C_{3t} where $\psi_0^{go}(C_{3t}) = 3$ and (ii) is attained in the case of the circulant graph $\Gamma = CR(5n, 2)$ and $k = -2$ where $\psi_{-2}^{go}(\Gamma) = 5$. For the case of the cube graph Q_3 bound (ii) is attained for $k = 2, 3$ where $\psi_2^{go}(Q_3) = \psi_3^{go}(Q_3) = 2$ and bound (iii) is attained for $k \in \{-2, -1\}$ where $\psi_{-2}^{go}(Q_3) = \psi_{-1}^{go}(Q_3) = 4$.

2.3. Bounding the cardinality of the sets belonging to a partition

In this section we obtain bounds for the cardinality of the sets belonging to a partition of a graph into global offensive k-alliances. To begin with, we present the following lemma, shown in [1], which is necessary for proving Theorem 14.

Lemma 13. *Let $\Gamma = (V, E)$ be a graph. If $S \subset V$ is a global offensive k-alliance, then $\overline{S}$ is a $\lfloor \frac{\Delta-k}{2} \rfloor$-dependent set.*

Theorem 14. *If S belongs to a partition of Γ into global offensive k-alliances, then* $\left\lceil \frac{n(2\delta-\Delta+k)}{\Delta+2\delta+k} \right\rceil \le |S| \le \left\lfloor \frac{2n\Delta}{\Delta+2\delta+k} \right\rfloor$.

Proof. If X is a t-dependent set in Γ, then for every $v \in X$, $\delta(v) - \delta_{\overline{X}}(v) \le t$. Thus,

$$\Delta(n - |X|) \ge \sum_{v \in \overline{X}} \delta_X(v) = \sum_{v \in X} \delta_{\overline{X}}(v) \ge \sum_{v \in X} (\delta - t) = |X|(\delta - t).$$

Hence,

$$|X| \le \frac{n\Delta}{\Delta + \delta - t}. \tag{2}$$

Now, if S belongs to a partition of Γ into global offensive k-alliances, then $\{S, \overline{S}\}$ is a partition of Γ into two global offensive k-alliances and, by Lemma 13, it is also a partition of $V(\Gamma)$ into two $\lfloor \frac{\Delta-k}{2} \rfloor$-dependent sets. Therefore, by taking $t = \lfloor \frac{\Delta-k}{2} \rfloor \le \frac{\Delta-k}{2}$ in (2) we obtain the upper bound on $|S|$. The lower bound on $|S|$ is deduced from the upper bound on $|\overline{S}| = n - |S|$. □

The circulant graph $CR(n, 2)$ contains a partition into two global offensive 0-alliances S and $\overline{S}$, such that $|S| = \lceil \frac{n}{3} \rceil$ and $|\overline{S}| = \lfloor \frac{2n}{3} \rfloor$, where the bounds of the above theorem are attained.

We recall that the Laplacian spectral radius of a graph Γ is defined as the largest eigenvalue of the Laplacian matrix of Γ.

Theorem 15. *Let $\Gamma = (V, E)$ be a graph with Laplacian spectral radius μ_*. If S belongs to a partition of Γ into global offensive k-alliances, $-\delta \le k \le \mu_* - \delta$, then*

$$\left\lceil \frac{n}{2} - \sqrt{\frac{n^2(\mu_* - k) - 2nm}{4\mu_*}} \right\rceil \le |S| \le \left\lfloor \frac{n}{2} + \sqrt{\frac{n^2(\mu_* - k) - 2nm}{4\mu_*}} \right\rfloor.$$

Proof. If S belongs to a partition of Γ into global offensive k-alliances, we know that S and $\overline{S}$ are global offensive k-alliances in Γ; then

$$\sum_{v \in \overline{S}} \delta(v) \le 2 \sum_{v \in \overline{S}} \delta_S(v) - k|\overline{S}|$$

and

$$\sum_{v \in S} \delta(v) \le 2 \sum_{v \in S} \delta_{\overline{S}}(v) - k|S|.$$

Therefore

$$2m \le 4 \sum_{v \in S} \delta_{\overline{S}}(v) - kn. \tag{3}$$

On the other hand, as was shown in [7, Theorem 6],

$$\mu_* \ge \frac{n \sum_{v \in S} \delta_{\overline{S}}(v)}{|S||\overline{S}|}. \tag{4}$$

Therefore, by using the expression (3) in (4) we obtain

$$\frac{2m + nk}{4} \le \frac{|S|(n - |S|)\mu_*}{n}.$$

By solving the above inequality for $|S|$ and considering that it is an integer we obtain the bounds on $|S|$. □

The above bounds are attained for the complete graph K_n for n even and $k = 1$. In this case K_n is partitioned into two global offensive 1-alliances of cardinality $\frac{n}{2}$.

2.4. The edge cut-set of a partition of Γ into global offensive k-alliances

Theorem 16. *Let Γ be a graph of order n and size m. If $C^{go}_{(r,k)}(\Gamma)$ is the minimum number of edges having its endpoints in different sets of a partition of Γ into $r \ge 2$ global offensive k-alliances, then*

(i) $C^{go}_{(r,k)}(\Gamma) \ge \left\lceil \frac{(r-1)(2m+nk)}{4} \right\rceil$,
(ii) *if* $r \ge 3$, *then* $C^{go}_{(r,k)}(\Gamma) \le \left\lfloor \frac{(r-1)(2m-nk)}{4(r-2)} \right\rfloor$,
(iii) *if* $r > 3$, *then* $C^{go}_{(r,k)}(\Gamma) \le \left\lfloor -\frac{nk(r-1)}{2r-6} \right\rfloor$.

Proof. Let $\Pi_r = \{S_1, S_2, \ldots, S_r\}$ be a partition of Γ into r global offensive k-alliances. The number of edges in Γ with one endpoint in S_i and the other endpoint in S_j is $C(S_i, S_j) = \sum_{v\in S_i} \delta_{S_j}(v) = \sum_{v\in S_j} \delta_{S_i}(v)$. Hence, taking into account that for every $v \in \overline{S_i}$, $\delta(v) \le 2\delta_{S_i}(v) - k$, we have

$$\begin{aligned}
2(r-1)m &= \sum_{i=1}^{r}\sum_{v\in\overline{S_i}}\delta(v) \le 2\sum_{i=1}^{r}\sum_{v\in\overline{S_i}}\delta_{S_i}(v) - k\sum_{i=1}^{r}(n-|S_i|)\\
&= 2\sum_{i=1}^{r}\sum_{j=1;j\ne i}^{r}\sum_{v\in S_j}\delta_{S_i}(v) - nk(r-1)\\
&= 2\sum_{i=1}^{r}\sum_{j=1;j\ne i}^{r} C(S_i,S_j) - nk(r-1)\\
&= 4C^{go}_{(r,k)}(\Gamma) - nk(r-1).
\end{aligned}$$

So, (i) follows. On the other hand if $r \ge 3$, since for every $v \in \overline{S_i}$, $\delta(v) \ge 2\delta_{\overline{S_i}}(v) + k$, we have

$$\begin{aligned}
2(r-1)m &= \sum_{i=1}^{r}\sum_{v\in\overline{S_i}}\delta(v) \ge 2\sum_{i=1}^{r}\sum_{v\in\overline{S_i}}\delta_{\overline{S_i}}(v) + k\sum_{i=1}^{r}(n-|S_i|)\\
&= 2\sum_{i=1}^{r}\sum_{j=1;j\ne i}^{r}\sum_{v\in S_j}\delta_{\overline{S_i}}(v) + nk(r-1)\\
&= 2\sum_{i=1}^{r}\sum_{j=1;j\ne i}^{r}\sum_{v\in S_j}\sum_{l=1;l\ne i}^{r}\delta_{S_l}(v) + nk(r-1)\\
&\ge 2\sum_{i=1}^{r}\sum_{j=1;j\ne i}^{r}\sum_{l=1;l\ne i,j}^{r}\sum_{v\in S_j}\delta_{S_l}(v) + nk(r-1)\\
&= 2\sum_{i=1}^{r}\sum_{j=1;j\ne i}^{r}\sum_{l=1;l\ne i,j}^{r} C(S_l,S_j) + nk(r-1)\\
&= 4(r-2)C^{go}_{(r,k)}(\Gamma) + nk(r-1).
\end{aligned}$$

Therefore, (ii) follows. Finally, as each S_i is a global offensive k-alliance, we have

$$\sum_{i=1}^{r}\sum_{v\in\overline{S_i}}\delta_{S_i}(v) \ge \sum_{i=1}^{r}\sum_{v\in\overline{S_i}}\delta_{\overline{S_i}}(v) + kn(r-1).$$

Hence, from the proof of (i) we have $2C^{go}_{(r,k)}(\Gamma) = \sum_{i=1}^{r}\sum_{v\in\overline{S_i}}\delta_{S_i}(v)$ and, from the proof of (ii), we have $\sum_{i=1}^{r}\sum_{v\in\overline{S_i}}\delta_{\overline{S_i}}(v) \ge 2(r-2)C^{go}_{(r,k)}(\Gamma)$. Therefore, we obtain $2C^{go}_{(r,k)}(\Gamma) \ge 2(r-2)C^{go}_{(r,k)}(\Gamma) + nk(r-1)$. The proof is complete. □

From the above result we have that if $\psi^{go}_k(\Gamma) \ge 3$ then $\psi^{go}_k(\Gamma) \le \left\lfloor \frac{6m+nk}{2m+nk} \right\rfloor$. Also, notice that, for $k \le \delta$, $2 \le \left\lfloor \frac{6m+nk}{2m+nk} \right\rfloor$, so we obtain the following bound on $\psi^{go}_k(\Gamma)$.

Corollary 17. *For any graph Γ of order n and size m, $\psi^{go}_k(\Gamma) \le \left\lfloor \frac{6m+nk}{2m+nk} \right\rfloor$.*

The above bound is attained, for instance, for the circulant graph $CR(5n, 2)$, where $\psi^{go}_{-2}(\Gamma) = 5$.

3. Partitioning $\Gamma_1 \times \Gamma_2$ into (global) offensive k-alliances

In [1] we can find the following result.

Theorem 18. *Let $\Gamma_i = (V_i, E_i)$ be a graph of minimum degree δ_i and maximum degree Δ_i, $i \in \{1, 2\}$. If S_i is an offensive k_i-alliance in Γ_i, $i \in \{1, 2\}$, then, for $k = \min\{k_2 - \Delta_1, k_1 - \Delta_2\}$, $S_1 \times S_2$ is an offensive k-alliance in $\Gamma_1 \times \Gamma_2$.*

From the above result we deduce that a partition

$$\Pi_{r_i}(\Gamma_i) = \{S^{(i)}_1, S^{(i)}_2, \ldots, S^{(i)}_{r_i}\}$$

of Γ_i into r_i offensive k_i-alliances, $i \in \{1, 2\}$, induces a partition of $\Gamma_1 \times \Gamma_2$ into r_1r_2 offensive k-alliances, with $k = \min\{k_2 - \Delta_1, k_1 - \Delta_2\}$:

$$\Pi_{r_1r_2}(\Gamma_1 \times \Gamma_2) = \left\{\begin{matrix} S_1^{(1)} \times S_1^{(2)} & \cdots & S_1^{(1)} \times S_{r_2}^{(2)} \\ S_2^{(1)} \times S_1^{(2)} & \cdots & S_2^{(1)} \times S_{r_2}^{(2)} \\ \vdots & \vdots & \vdots \\ S_{r_1}^{(1)} \times S_1^{(2)} & \cdots & S_{r_1}^{(1)} \times S_{r_2}^{(2)} \end{matrix}\right\}.$$

So, we obtain the following result.

Corollary 19. *For any graph Γ_i of maximum degree Δ_i, $i \in \{1, 2\}$, and for every $k \le \min\{k_1 - \Delta_2, k_2 - \Delta_1\}$, $\psi_k^o(\Gamma_1 \times \Gamma_2) \ge \psi_{k_1}^o(\Gamma_1)\psi_{k_2}^o(\Gamma_2)$.*

For the particular case of the graph $C_4 \times K_4$, we have $\psi_{-3}^o(C_4 \times K_4) = 8 = 4 \cdot 2 = \psi_0^o(C_4)\psi_1^o(K_4)$.

Theorem 20. *Let $\Gamma_i = (V_i, E_i)$ be a graph of order n_i and let $\Pi_{r_i}(\Gamma_i)$ be a partition of Γ_i into r_i global offensive k_i-alliances, $i \in \{1, 2\}$. If $x_i = \min_{S \in \Pi_{r_i}(\Gamma_i)}\{|S|\}$ and $k \le \min\{k_1, k_2\}$, then*

(i) $\gamma_k^o(\Gamma_1 \times \Gamma_2) \le \min\{n_2x_1, n_1x_2\}$,
(ii) $\psi_k^{go}(\Gamma_1 \times \Gamma_2) \ge \max\{\psi_{k_1}^{go}(\Gamma_1), \psi_{k_2}^{go}(\Gamma_2)\}$.

Proof. If we consider the set $M_j = S_j^{(1)} \times V_2$ where $S_j^{(1)} \in \Pi_{r_1}(\Gamma_1)$, for every $(u, v) \notin M_j$ it is satisfied that

$$\delta_{S_j^{(1)} \times V_2}(u, v) = \delta_{S_j^{(1)}}(u) \ge \delta_{\overline{S_j^{(1)}}}(u) + k_1 = \delta_{\overline{S_j^{(1)} \times V_2}}(u, v) + k_1,$$

and thus M_j is a global offensive k_1-alliance in $\Gamma_1 \times \Gamma_2$. The same argument shows that $N_l = V_1 \times S_l^{(2)}$ is a global offensive k_2-alliance for every $S_l^{(2)} \in \Pi_{r_2}(\Gamma_2)$. Thus, by taking $S_j^{(1)}$ and $S_l^{(2)}$ of cardinality x_1 and x_2, respectively, we obtain $|M_j| = x_1n_2$ and $|N_l| = x_2n_1$, so (i) follows. Moreover, as $\{M_1, \ldots, M_{r_1}\}$ and $\{N_1, \ldots, N_{r_2}\}$ are partitions of $\Gamma_1 \times \Gamma_2$ into global offensive k-alliances, (ii) follows. □

Suppose Γ_j is partitionable into global offensive k_j-alliances, for $k_j \ge 1$ and $j \in \{1, 2\}$. Bound (ii) is attained for $1 \le k \le \min\{k_1, k_2\}$, where $\psi_k^{go}(\Gamma_1 \times \Gamma_2) = 2 = \max\{2, 2\} = \max\{\psi_{k_1}^{go}(\Gamma_1), \psi_{k_2}^{go}(\Gamma_2)\}$. From (ii) we deduce the following result.

Corollary 21. *If a graph Γ_i of order n_i is partitionable into global offensive k_i-alliances, $i \in \{1, 2\}$, then for $k \le \min\{k_1, k_2\}$,*

$$\gamma_k^o(\Gamma_1 \times \Gamma_2) \le \frac{n_1n_2}{\max\{\psi_{k_1}^{go}(\Gamma_1), \psi_{k_2}^{go}(\Gamma_2)\}}.$$

An example with equality is $\gamma_1^o(C_4 \times K_2) = \frac{4 \cdot 2}{\max\{\psi_1^{go}(C_4), \psi_1^{go}(K_2)\}} = 4$.

Acknowledgements

This work was partly supported by the Spanish Ministry of Science and Innovation through projects TSI2007-65406-C03-01 "E-AEGIS", CONSOLIDER INGENIO 2010 CSD2007-0004 "ARES", MTM2009-07800 and MTM2009-09501, and by the Junta de Andalucía, ref. FQM-260 and ref. P06-FQM-02225.

Appendix

The global offensive k-alliance partition number of the circulant graph $CR(n, 2)$ has been used throughout the article. In this appendix we compute it. To begin with, we emphasize the following two obvious claims.

Claim 22. *For the circulant graph $\Gamma = CR(n, 2)$, $\gamma(\Gamma) = \lceil \frac{n}{5} \rceil$.*

Claim 23. *Any dominating set in $\Gamma = CR(n, 2)$ is a global offensive (-2)-alliance.*

Remark 24. In the case of the circulant graph $\Gamma = CR(n, 2)$ we have the following:

(i) Γ is not partitionable into global offensive 3-alliances or global offensive 4-alliances.
(ii) $\psi_1^{go}(\Gamma) = \psi_2^{go}(\Gamma) = 2$ if and only if $n = 4j$.
(iii) $\psi_{-1}^{go}(\Gamma) = \psi_0^{go}(\Gamma) = 3$ if and only if $n = 3j$.
(iv) $\psi_{-2}^{go}(\Gamma) = \left\lfloor \frac{n}{\lceil \frac{n}{5} \rceil} \right\rfloor$.

Proof. We first emphasize that, since Γ is a 4-regular graph, $\psi^{go}_{-1}(\Gamma) = \psi^{go}_{0}(\Gamma)$, $\psi^{go}_{1}(\Gamma) = \psi^{go}_{2}(\Gamma)$ and $\psi^{go}_{3}(\Gamma) = \psi^{go}_{4}(\Gamma)$. So we will only consider the study of $\psi^{go}_{k}(\Gamma)$ for $k = 4, 2, 0, -2$. Let us denote the vertices of Γ by $\{v_1, v_2, \ldots, v_n\}$ such that v_i is adjacent to $v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}$.

(i) By Corollary 4 we know that if Γ is partitionable into global offensive k-alliances for $k \geq 1$, then $\psi^{go}_{k}(\Gamma) = 2$. So we suppose $\{S_1, S_2\}$ is a partition of the graph into two global offensive 4-alliances. If $v_i \notin S_j$, then $\delta_{S_j}(v_i) = 4 = \delta(v_i)$, so $CR(n, 2)$ is a bipartite graph, a contradiction.

(ii) As above, let us suppose $\{S_1, S_2\}$ is a partition of the graph into two global offensive 2-alliances. If $v_i \notin S_1$, then $\delta_{S_1}(v_i) \geq 3$. If $\delta_{S_1}(v_i) = 4$, then $\delta_{S_1}(v_{i+1}) \geq 2$, so $\delta_{S_2}(v_{i+1}) \leq 2 < \delta_{\overline{S_2}}(v_{i+1}) + 2$, a contradiction. Thus $\delta_{S_1}(v_i) = 3$. Analogously for S_2, if $v_i \notin S_2$, then $\delta_{S_2}(v_i) = 3$.

Now, let $v_i \in S_2$; if $v_{i-2}, v_{i-1}, v_{i+1} \in S_1$ (or $v_{i-1}, v_{i+1}, v_{i+2} \in S_1$), we obtain that $\delta_{S_2}(v_{i-1}) \leq 2$ (or $\delta_{S_2}(v_{i+1}) \leq 2$), which is a contradiction. Therefore, if $v_l, v_{l+1} \in S_1$, then necessarily, $v_{l+2}, v_{l+3} \in S_2$ and this is possible if and only if $n = 4j$.

(iii) Let us suppose $n = 3j$. So, the sets $\{v_1, v_4, \ldots, v_{n-2}\}$, $\{v_2, v_5, \ldots, v_{n-1}\}$ and $\{v_3, v_6, \ldots, v_n\}$ form a partition of Γ into three global offensive 0-alliances; therefore $\psi^{go}_{0}(\Gamma) \geq 3$. From Corollary 5, we have that $\psi^{go}_{0}(\Gamma) \leq 3$, so $\psi^{go}_{0}(\Gamma) = 3$. On the contrary, let us suppose $\psi^{go}_{0}(\Gamma) = 3$; then by Theorem 6 and Remark 8 each alliance of the partition is a maximal independent set and the chromatic number of Γ is 3, so there exist three color classes among the vertices of Γ, $\dot{v_1}$, $\dot{v_2}$ and $\dot{v_3}$, which contain those vertices with subindexes congruent to 1, 2 and 3, respectively, and hence v_n belongs to the class $\dot{v_3}$.

(iv) We have that $\psi^{go}_{-2}(\Gamma)\gamma^{go}_{-2}(\Gamma) \leq n$; now, by using Claims 22 and 23, we obtain $\psi^{go}_{-2}(\Gamma) \leq \left\lfloor \frac{n}{\lceil \frac{n}{5} \rceil} \right\rfloor$. By taking $q = \left\lfloor \frac{n}{\lceil \frac{n}{5} \rceil} \right\rfloor$, let us form a partition of the graph into q dominating sets. Note that $2 < q \leq 5$. Hence, we have the following cases:

Case 1: $q = 5$ if and only if $n = 5j$, $j \in \mathbb{Z}_+$. The sets $\{v_1, v_6, \ldots, v_{n-4}\}$, $\{v_2, v_7, \ldots, v_{n-3}\}$, $\{v_3, v_8, \ldots, v_{n-2}\}$, $\{v_4, v_9, \ldots, v_{n-1}\}$ and $\{v_5, v_{10}, \ldots, v_n\}$ form a partition of Γ into five dominating sets.

Case 2: $q = 4$ if and only if $n \neq 6, 7, 11, 5j, j \in Z_+$. So, if $n = 4j + r$, $r \in \{0, 1, 2, 3\}$, then P_r is a partition of Γ into dominating sets:

$$P_0 = \{\{v_1, v_5, \ldots, v_{n-3}\}, \{v_2, v_6, \ldots, v_{n-2}\}, \{v_3, v_7, \ldots, v_{n-1}\}, \{v_4, v_8, \ldots, v_n\}\},$$

$$\begin{aligned} P_1 &= \{\{v_1, v_6, v_{10}, v_{14} \ldots, v_{n-3}\}, \{v_2, v_7, v_{11}, v_{15}, \ldots, v_{n-2}\}, \\ &\quad \{v_3, v_8, v_{12}, v_{16}, \ldots, v_{n-1}\}, \{v_4, v_5, v_9, v_{13}, \ldots, v_n\}\}, \\ P_2 &= \{\{v_1, v_6, v_{11}, v_{15}, v_{19}, \ldots, v_{n-3}\}, \{v_2, v_7, v_{12}, v_{16}, v_{20}, \ldots, v_{n-2}\}, \\ &\quad \{v_3, v_8, v_{13}, v_{17}, v_{21}, \ldots, v_{n-1}\}, \{v_4, v_5, v_9, v_{10}, v_{14}, v_{18}, \ldots, v_n\}\}, \\ P_3 &= \{\{v_1, v_6, v_{11}, v_{16}, v_{20}, v_{24}, \ldots, v_{n-3}\}, \{v_2, v_7, v_{12}, v_{17}, v_{21}, v_{25}, \ldots, v_{n-2}\}, \\ &\quad \{v_3, v_8, v_{13}, v_{18}, v_{22}, v_{26}, \ldots, v_{n-1}\}, \{v_4, v_5, v_9, v_{10}, v_{14}, v_{15}, v_{19}, v_{23}, \ldots, v_n\}\}. \end{aligned}$$

Case 3: $q = 3$ if and only if $n = 6, 7, 11$. In these cases, $P_6 = \{\{v_1, v_2\}, \{v_3, v_4\}, \{v_5, v_6\}\}$, $P_7 = \{\{v_1, v_4\}, \{v_2, v_5\}, \{v_3, v_6, v_7\}\}$ and $P_{11} = \{\{v_1, v_4, v_7, v_{11}\}, \{v_2, v_5, v_8, v_{10}\}, \{v_3, v_6, v_9\}\}$ form partitions into three dominating sets.

Therefore, by using Claim 23 we conclude the proof. □

References

[1] S. Bermudo, J.A. Rodríguez-Velázquez, J.M. Sigarreta, I.G. Yero, On global offensive k-alliances in graphs, Applied Mathematics Letters (2010) doi:10.1016/j.aml.2010.08.008.
[2] R.C. Brigham, R.D. Dutton, S.T. Hedetniemi, A sharp lower bound on the powerful alliance number of $C_m \times C_n$, Congressus Numerantium 167 (2004) 57–63.
[3] M. Chellali, T. Haynes, Global alliances and independence in trees, Discussiones Mathematicae Graph Theory 27 (1) (2007) 19–27.
[4] L. Eroh, R. Gera, Global alliance partition in trees, Journal of Combinatorial Mathematics and Combinatorial Computing 66 (2008) 161–169.
[5] L. Eroh, R. Gera, Alliance partition number in graphs, Ars Combinatoria (in press).
[6] O. Favaron, G. Fricke, W. Goddard, S.M. Hedetniemi, S.T. Hedetniemi, P. Kristiansen, R.C. Laskar, R.D. Skaggs, Offensive alliances in graphs, Discussiones Mathematicae Graph Theory 24 (2) (2004) 263–275.
[7] H. Fernau, J.A. Rodríguez, J.M. Sigarreta, Offensive k-alliances in graphs, Discrete Applied Mathematics 157 (1) (2009) 177–182.
[8] T.W. Haynes, S.T. Hedetniemi, M.A. Henning, Global defensive alliances in graphs, The Electronic Journal of Combinatorics 10 (2003) 139–146.
[9] T.W. Haynes, J.A. Lachniet, The alliance partition number of grid graphs, AKCE International Journal of Graphs and Combinatorics 4 (1) (2007) 51–59.
[10] P. Kristiansen, S.M. Hedetniemi, S.T. Hedetniemi, Alliances in graphs, Journal of Combinatorial Mathematics and Combinatorial Computing 48 (2004) 157–177.
[11] J.A. Rodríguez-Velázquez, I.G. Yero, J.M. Sigarreta, Defensive k-alliances in graphs, Applied Mathematics Letters 22 (2009) 96–100.
[12] J.A. Rodríguez-Velázquez, J.M. Sigarreta, Global defensive k-alliances in graphs, Discrete Applied Mathematics 157 (2) (2009) 211–218.
[13] K.H. Shafique, R.D. Dutton, Maximum alliance-free and minimum alliance-cover sets, Congressus Numerantium 162 (2003) 139–146.
[14] K.H. Shafique, Partitioning a graph in alliances and its application to data clustering, Ph. D. Thesis, 2004.
[15] K.H. Shafique, R.D. Dutton, On satisfactory partitioning of graphs, Congressus Numerantium 154 (2002) 183–194.
[16] J.M. Sigarreta, S. Bermudo, H. Fernau, On the complement graph and defensive k-alliances, Discrete Applied Mathematics 157 (8) (2009) 1687–1695.
[17] J.M. Sigarreta, J.A. Rodríguez, On defensive alliance and line graphs, Applied Mathematics Letters 19 (12) (2006) 1345–1350.
[18] J.M. Sigarreta, J.A. Rodríguez, On the global offensive alliance number of a graph, Discrete Applied Mathematics 157 (2) (2009) 219–226.